

Comments on Open Wilson Lines and Generalized Star Products

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We consider an open Wilson line as a momentum representation of a boundary state which describes a D -brane in a constant B -field background. Using this picture, we study the Seiberg-Witten map which relates the commutative and noncommutative gauge fields, and determine the products of fields appearing in the general terms in the expansion of this map.

1. Introduction

In the recent study of noncommutative gauge theories, the open Wilson lines play an important role. They are used to construct gauge invariant operators [1,2,3,4], and also appear in the couplings between open and closed string modes [5,6,7,8,9]. The correlation functions of open Wilson lines are studied in [10,11,12].

In the expansion of an open Wilson line as a power series of the noncommutative gauge field, there appears the so-called generalized star products [13]. These products also appear in the scattering amplitudes of string theories and noncommutative field theories [14,15,16,17,18]. In [19,13], open Wilson lines were used to construct the Seiberg-Witten map which relates the commutative and noncommutative gauge fields [20]. In this map, there also appears the structure of the generalized star products.

In this short note, we consider an open Wilson line as a momentum representation of a boundary state describing a D -brane in a constant B -field background. Using this picture, we study the Seiberg-Witten map and determine the products of fields which appear in the general terms in the expansion of this map.

This paper is organized as follows: In section 2, we review the open Wilson lines from the viewpoint of boundary state. In section 3, we show that the generalized star products can be written as correlation functions on \mathbf{S}^1 . In section 4, we construct the Seiberg-Witten map up to $\mathcal{O}(A^3)$ from the gauge equivalence condition and compare this result with that obtained from the open Wilson line. Section 5 is devoted to the conclusion and discussion.

2. Open Wilson Lines from Boundary State

In this section, we review how open contours appear in gauge invariant objects in noncommutative gauge theories [1,4] from the viewpoint of boundary state. Let us consider a gauge theory on noncommutative \mathbf{R}^n , which is defined by the algebra

$$[x^i, x^j] = i\theta^{ij}. \quad (2.1)$$

On this space, the open Wilson line $W(k)$ is defined by

$$W(k) = \text{Tr} \exp \left(-ik_i (x^i + \theta^{ij} \hat{A}_j) \right)_\star \quad (2.2)$$

where \hat{A} is the noncommutative gauge field and the product of fields is taken by the star product

$$f \star g = f \exp \left(\frac{i}{2} \overleftarrow{\partial}_k \theta^{kl} \overrightarrow{\partial}_l \right) g. \quad (2.3)$$

In the boundary state formalism, $W(k)$ can be written as an overlap between the closed string tachyon $|k\rangle$ and the boundary state $|B\rangle$ describing a D -brane in a constant B -field background [21]:

$$W(k) = \langle k|B\rangle = \int [dx] \exp \left(i \int_0^1 d\sigma \frac{1}{2} x^i B_{ij} \partial_\sigma x^j - k_i (x^i + \theta^{ij} \hat{A}_j) \right). \quad (2.4)$$

θ and B are related by

$$\theta = \frac{1}{B}. \quad (2.5)$$

The closed string configuration $x(\sigma)$ is parametrized by $\sigma \in [0, 1]$.

We can consider the general Wilson line which is the overlap between $|B\rangle$ and the closed string momentum eigenstate:

$$\begin{aligned} \langle P(\sigma)|B\rangle &= \text{Tr } \mathbf{P} \exp \left(-i \int_0^1 d\sigma P_i(\sigma) (x^i + \theta^{ij} \hat{A}_j) \right)_* \\ &= \int [dx] \exp \left(i \int_0^1 d\sigma \frac{1}{2} x^i B_{ij} \partial_\sigma x^j - P_i (x^i + \theta^{ij} \hat{A}_j) \right), \end{aligned} \quad (2.6)$$

where \mathbf{P} denotes the path ordering.

At first sight, it seems impossible to obtain the open contour from the boundary state which is an element of the Hilbert space of a closed string. As shown in [1,4], the open contour does appear when we write the exponential in (2.2) as a product of small segments and move the factor e^{-ikx} to the leftmost. In the rest of this section, we shall rederive this result from the path integral formalism. To do this, we should carefully treat the boundary condition of the integration variable $x(\sigma)$. First we write $\langle P(\sigma)|B\rangle$ as

$$\begin{aligned} \langle P(\sigma)|B\rangle &= \int dx_0 \langle x_0 | \mathbf{P} e^{-i \int_0^1 d\sigma P(\sigma)(x + \theta \hat{A})} | x_0 \rangle \\ &= \int dx_0 \langle x_0 + y(1) | e^{-i\Pi(1)y(1)} \mathbf{P} e^{-i \int_0^1 d\sigma P(\sigma)(x + \theta \hat{A})} e^{i\Pi(0)y(0)} | x_0 + y(0) \rangle \end{aligned} \quad (2.7)$$

where Π is the conjugate momentum of x . Here we introduced an arbitrary path $y(\sigma)$ which may or may not be open at this stage. We can rewrite (2.7) as a phase space path integral with constraints:

$$\begin{aligned} \langle P(\sigma)|B\rangle &= \int [d\Pi dx_{\text{open}}] \delta(\chi_i) \det^{\frac{1}{2}} \{ \chi_i, \chi_j \} \\ &\quad \cdot \exp \left(i \int_0^1 d\sigma \Pi_i \partial_\sigma x_{\text{open}}^i - P_i (x_{\text{open}}^i + \theta^{ij} \hat{A}_j) - \partial_\sigma (\Pi_i y^i) \right). \end{aligned} \quad (2.8)$$

The constraints are given by

$$\chi_i = \Pi_i + \frac{1}{2} B_{ij} x_{\text{open}}^j, \quad (2.9)$$

which are the second class: $\{\chi_i, \chi_j\} = B_{ij}$. The boundary condition of x_{open} is specified by

$$x_{\text{open}}(1) - x_{\text{open}}(0) = y(1) - y(0). \quad (2.10)$$

Integrating out Π , $\langle P(\sigma)|B \rangle$ becomes

$$\begin{aligned} \langle P(\sigma)|B \rangle = \int [dx_{\text{open}}] \exp \left(i \int_0^1 d\sigma \frac{1}{2} x_{\text{open}}^i B_{ij} \partial_\sigma x_{\text{open}}^j - P_i(x_{\text{open}}^i + \theta^{ij} \hat{A}_j) \right. \\ \left. + \frac{i}{2} \partial_\sigma (y^i B_{ij} x_{\text{open}}^j) \right). \end{aligned} \quad (2.11)$$

In this expression, the contour $y(\sigma)$ is arbitrary. In the case $\partial_\sigma y = \theta P$, by shifting the variable x_{open} to $x + y$, $\langle P(\sigma)|B \rangle$ can be written as

$$\begin{aligned} \langle P(\sigma)|B \rangle = \int_{\text{periodic}} [dx] \exp \left(i \int_0^1 d\sigma \frac{1}{2} x^i B_{ij} \partial_\sigma x^j - \frac{1}{2} y^i B_{ij} \partial_\sigma y^j \right. \\ \left. + \partial_\sigma (y^i B_{ij} x^j) + \partial_\sigma y^i \hat{A}_i(x + y) \right) \\ = e^{-i \int_0^1 \frac{1}{2} y^i B_{ij} \partial_\sigma y^j} \text{Tr} e^{-ipx} \mathbf{P} \exp \left(i \int_0^1 d\sigma \partial_\sigma y^i \hat{A}_i(x + y) \right). \end{aligned} \quad (2.12)$$

In the last step, we used the relation

$$y(1) - y(0) = \int_0^1 \partial_\sigma y = \theta p, \quad (2.13)$$

where p is the center of mass momentum of closed string

$$p = \int_0^1 d\sigma P(\sigma). \quad (2.14)$$

In summary, the open contour appears in the y -space which is defined by

$$\partial_\sigma y^i(\sigma) = \theta^{ij} P_j(\sigma). \quad (2.15)$$

This is the same relation as the T-duality except the factor of θ . Therefore, we can understand the appearance of open contour as the winding mode which is T-dual to the center of mass momentum. Note that the straight open line corresponds to the momentum $P(\sigma)$ without non-zero modes, or the closed string tachyon $|k\rangle$ (see [9] for the recent discussion).

3. Generalized Star Product

In [13,19], it was shown that the generalized star products appears when we expand the open Wilson lines. In the path integral representation, the generalized star product can be written as a correlation function of exponential operators on \mathbf{S}^1 :

$$\delta(k - \sum_a k_a) J_n(k_1, \dots, k_n) = \left\langle e^{-i \int d\sigma k x(\sigma)} \prod_{a=1}^n \int_0^1 d\sigma_a e^{i k_a x(\sigma_a)} \right\rangle. \quad (3.1)$$

Here, the expectation value is defined by

$$\langle \mathcal{O} \rangle = \int_{\text{periodic}} [dx] \mathcal{O} \exp \left(i \int_0^1 d\sigma \frac{1}{2} x^i B_{ij} \partial_\sigma x^j \right). \quad (3.2)$$

The factor $\delta(k - \sum_a k_a)$ in (3.1) comes from the integral over the zero-mode of $x(\sigma)$. The non-zero mode $\tilde{x}(\sigma)$ of $x(\sigma)$ is defined by

$$x(\sigma) = x_0 + \tilde{x}(\sigma), \quad \int_0^1 d\sigma \tilde{x}(\sigma) = 0. \quad (3.3)$$

The propagator of non-zero mode of x on \mathbf{S}^1 should satisfy

$$-i B_{ik} \partial_1 \langle \tilde{x}^k(\sigma_1) \tilde{x}^j(\sigma_2) \rangle = \delta_i^j \left[\delta(\sigma_1, \sigma_2) - 1 \right]. \quad (3.4)$$

The solution of this condition is

$$\langle \tilde{x}(\sigma_1) \tilde{x}(\sigma_2) \rangle = -\frac{i}{2} \theta \tau(\sigma_{12}) \quad (3.5)$$

where $\sigma_{12} = \sigma_1 - \sigma_2$ and

$$\tau(\sigma) = 2\sigma - \epsilon(\sigma). \quad (3.6)$$

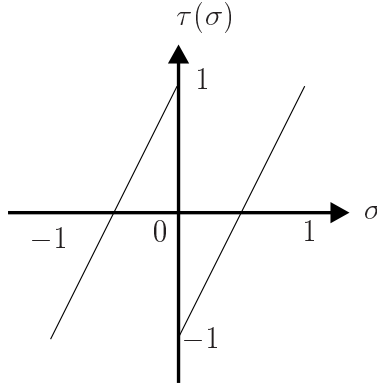


Fig. 1: Graph of $\tau(\sigma)$

$\tau(\sigma)$ in (3.6) is defined on the region $-1 \leq \sigma \leq 1$. We can extend $\tau(\sigma)$ to the outside of this region with periodicity 1. As a Fourier series, $\tau(\sigma)$ can be written as

$$\tau(\sigma) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin(2\pi n \sigma). \quad (3.7)$$

J_n in (3.1) is the correlation function of non-zero modes, which is written as [13]

$$J_n(k_1, \dots, k_n) = \prod_{a=1}^n \int_0^1 d\sigma_a \exp \left(\frac{i}{2} \sum_{a < b} k_a \theta k_b \tau(\sigma_{ab}) \right). \quad (3.8)$$

We denote the product determined by J_n as $(f_1, \dots, f_n)_n$. Note that this product is totally symmetric.

By inserting

$$1 = \prod_{a=1}^{n-1} \int_{-1}^1 ds_{an} \delta(s_{an} - \tau(\sigma_{an})) \quad (3.9)$$

and integrating over σ_a ($a = 1, \dots, n-1$), J_n can also be written as

$$J_n = \frac{1}{2^{n-1}} \prod_{a=1}^{n-1} \int_{-1}^1 ds_{an} \exp \left(\frac{i}{2} \sum_{a < b} k_a \theta k_b s_{ab} \right) \quad (3.10)$$

where s_{ab} ($a, b \neq n$) is given by

$$s_{ab} = \tau \left(\frac{s_{an} - s_{bn}}{2} \right). \quad (3.11)$$

J_n is known to satisfy the following descent relation [13]

$$\frac{1}{2} \sum_{a=1}^n k_a \theta k_n J_n(k_1, \dots, k_n) = \sum_{a=1}^{n-1} \sin \left(\frac{1}{2} k_a \theta k_n \right) J_{n-1}(k_1, \dots, k_a + k_n, \dots, k_{n-1}). \quad (3.12)$$

In the position space, this relation can be written as (*e.g.* for $n = 2, 3$)

$$\begin{aligned} i\theta^{kl}(\partial_k, \partial_l g) &= [f, g]_{\star}, \\ i\theta^{kl}\partial_k(f, g, \partial_l h)_3 &= (f, [g, h]_{\star}) + (g, [f, h]_{\star}), \end{aligned} \quad (3.13)$$

where $[f, g]_{\star} = f \star g - g \star f$ and $(f, g) \equiv (f, g)_2$.

4. Seiberg-Witten Map and Generalized Star Product

In this section, we consider the Seiberg-Witten map from the viewpoint of the gauge equivalence condition [20] and the two descriptions of an open Wilson line [21,22].

4.1. Seiberg-Witten Map from Gauge Equivalence Condition

First we consider the Seiberg-Witten map as the correspondence between commutative and noncommutative gauge transformation. To construct the map, it is more convenient to see this map as the correspondence of two BRST transformations. The nilpotent BRST transformation associated with commutative gauge transformation is

$$\delta A_i = \partial_i c, \quad \delta c = 0. \quad (4.1)$$

For the noncommutative case, the BRST transformation is given by

$$\delta \hat{A}_i = \partial_i \hat{c} - i[\hat{A}_i, \hat{c}]_\star, \quad \delta \hat{c} = \frac{i}{2}[\hat{c}, \hat{c}]_\star. \quad (4.2)$$

c and \hat{c} are ghost fields for commutative and noncommutative gauge symmetries, respectively. Note that $[\hat{c}, \hat{c}]_\star$ does not vanish since \hat{c} is a fermionic variable.

We can construct the Seiberg-Witten map as a power series in A :

$$\begin{aligned} \hat{A} &= A + \sum_{n=2}^{\infty} A^{(n)} = \hat{A}(A), \\ \hat{c} &= c + \sum_{n=1}^{\infty} c^{(n)} = \hat{c}(c, A), \end{aligned} \quad (4.3)$$

where the superscript n denotes the order of A . First, let us consider the map for ghosts. By equating the terms of the same order in A , the relation $\delta \hat{c} = i[\hat{c}, \hat{c}]_\star/2$ can be decomposed as

$$\begin{aligned} \delta c^{(1)} &= \frac{i}{2}[c, c]_\star = -\frac{1}{2}\theta^{kl}(\partial_k c, \partial_l c), \\ \delta c^{(2)} &= i[c, c^{(1)}]_\star. \end{aligned} \quad (4.4)$$

Here, we used the relation (3.13). By “integrating” this relation, we find that $c^{(1)}$ and $c^{(2)}$ are given by

$$\begin{aligned} c^{(1)} &= -\frac{1}{2}\theta^{kl}(A_k, \partial_l c), \\ c^{(2)} &= -\frac{1}{2}\theta^{kl}\theta^{mn}\langle A_k, \partial_m c, \partial_l A_n \rangle, \end{aligned} \quad (4.5)$$

where $\langle \rangle$ is defined by

$$\langle f, g, h \rangle = (f, (g, h)) + (g, (f, h)) - (f, g, h)_3. \quad (4.6)$$

This product is symmetric in the first two arguments: $\langle f, g, h \rangle = \langle g, f, h \rangle$. Now we consider the map for gauge fields. The equations satisfied by $A^{(2)}$ and $A^{(3)}$ are

$$\begin{aligned}\delta A_i^{(2)} &= \partial_i c^{(1)} - i[A_i, c]_\star, \\ \delta A_i^{(3)} &= \partial_i c^{(2)} - i[A_i^{(2)}, c]_\star - i[A_i, c^{(1)}]_\star.\end{aligned}\tag{4.7}$$

The solution for these conditions is found to be

$$\begin{aligned}A_i^{(2)} &= -\frac{1}{2}\theta^{kl}(A_k, \partial_l A_i + F_{li}), \\ A_i^{(3)} &= -\frac{1}{2}\theta^{kl}\theta^{mn}\left[\langle A_k, \partial_m A_i, \partial_l A_n \rangle + \langle A_k, F_{mi}, F_{ln} \rangle + \langle A_k, A_m, \partial_l F_{in} \rangle\right].\end{aligned}\tag{4.8}$$

4.2. Path Integral Derivation of Seiberg-Witten Map

In this subsection, we derive the Seiberg-Witten map from the open Wilson line in commutative and noncommutative pictures. We will see that the map obtained by this method agrees with that in the previous subsection. The relation between A and \hat{A} can be written as

$$\widehat{W}(\hat{A}) = W(A)\tag{4.9}$$

where

$$\begin{aligned}\widehat{W}(\hat{A}) &= \left\langle \exp\left(-ipx(0) + i\int_0^1 d\sigma \hat{A}_i(x+y)\partial_\sigma y^i\right) \right\rangle, \\ W(A) &= \left\langle \exp\left(-ipx(0) + i\int_0^1 d\sigma A_i(x+y)\partial_\sigma(x^i + y^i)\right) \right\rangle.\end{aligned}\tag{4.10}$$

As discussed in [22], Seiberg-Witten map can be extracted from (4.9) by expanding the exponential and equating the term with single σ -integral. The term with single σ -integral in $\widehat{W}(\hat{A})$ is

$$i\int_0^1 d\sigma \partial_\sigma y^i \hat{A}_i(p) e^{ipy(\sigma)}.\tag{4.11}$$

The corresponding term in $W(A)$ has the form

$$i\int_0^1 d\sigma \partial_\sigma y^i \sum_{n=1}^{\infty} A_i^{(n)}(p) e^{ipy(\sigma)}\tag{4.12}$$

where $A^{(n)}$ is the n -th order term of A . Let us compute the first three terms of this expansion. As easily seen, $A^{(1)}$ is equal to A . $A^{(2)}$ is found to be

$$A_i^{(2)}(p) = -\frac{1}{2}\theta^{kl} \int dk_1 dk_2 \delta(p - k_1 - k_2) I_2(k_1, k_2) A_k(k_1) (\partial_l A_i + F_{li})(k_2)\tag{4.13}$$

where I_2 is given by

$$I_2(k_1, k_2) = \int_0^1 d\sigma_1 \delta(\sigma_{12}) \exp\left(\frac{i}{2} k_1 \theta k_2 \tau(\sigma_{12})\right). \quad (4.14)$$

The factor $\delta(\sigma_{12})$ comes from the contraction between $\partial_1 x(\sigma_1)$ and $x(\sigma_2)$. By regularizing the δ -function as in [22], we find that I_2 is equal to J_2 :

$$I_2(k_1, k_2) = \frac{1}{2} \int_{-1}^1 d\epsilon_{12} \exp\left(-\frac{i}{2} k_1 \theta k_2 \epsilon_{12}\right) = J_2(k_1, k_2). \quad (4.15)$$

Therefore, (4.13) reproduces the result (4.8) obtained from the gauge equivalence relation.

Let us consider $A^{(3)}$. From (4.10), $A^{(3)}$ is found to be

$$\begin{aligned} A_i^{(3)}(p) = & -\frac{1}{2} \theta^{kl} \theta^{mn} \int \prod_{a=1}^3 dk_a \delta(p - \sum_a k_a) I_3(k_1, k_2, k_3) \\ & \cdot A_k(k_1) \left[\partial_m A_i(k_2) \partial_l A_n(k_3) + F_{mi}(k_2) F_{ln}(k_3) + A_m(k_2) \partial_l F_{in}(k_3) \right] \end{aligned} \quad (4.16)$$

where I_3 is given by

$$\begin{aligned} I_3 &= \int_0^1 d\sigma_1 d\sigma_2 \delta(\sigma_{13}) \delta(\sigma_{23}) \exp\left(\frac{i}{2} \sum_{a<b}^3 k_a \theta k_b \tau(\sigma_{ab})\right) \\ &= \prod_{a=1}^3 \int_0^1 d\sigma_a \delta(\sigma_{13}) \delta(\sigma_{23}) \exp\left(\frac{i}{2} \sum_{a<b}^3 k_a \theta k_b \tau(\sigma_{ab})\right). \end{aligned} \quad (4.17)$$

Here we put an extra integral over σ_3 since I_3 does not depend on σ_3 . To evaluate I_3 , the following identity is useful:

$$\frac{1}{4} \prod_{a=1}^3 \int_0^1 d\sigma_a \partial_1 \tau(\sigma_{13}) \partial_2 \tau(\sigma_{23}) \exp\left(\frac{i}{2} \sum_{a<b}^3 k_a \theta k_b \tau(\sigma_{ab})\right) = 0. \quad (4.18)$$

This identity holds due to the periodicity of $\tau(\sigma)$. Using the relation $\frac{1}{2} \partial_\sigma \tau(\sigma) = 1 - \delta(\sigma)$, I_3 can be written as

$$\begin{aligned} I_3 &= \prod_{a=1}^3 \int_0^1 d\sigma_a \left[\delta(\sigma_{13}) \delta(\sigma_{23}) - (1 - \delta(\sigma_{13}))(1 - \delta(\sigma_{23})) \right] \exp\left(\frac{i}{2} \sum_{a<b}^3 k_a \theta k_b \tau(\sigma_{ab})\right) \\ &= \prod_{a=1}^3 \int_0^1 d\sigma_a \left[\delta(\sigma_{13}) + \delta(\sigma_{23}) - 1 \right] \exp\left(\frac{i}{2} \sum_{a<b}^3 k_a \theta k_b \tau(\sigma_{ab})\right) \\ &= J_2(k_1, k_3) J_2(k_1 + k_3, k_2) + J_2(k_2, k_3) J_2(k_1, k_2 + k_3) - J_3(k_1, k_2, k_3). \end{aligned} \quad (4.19)$$

We can see that the product defined by I_3 is nothing but $\langle f, g, h \rangle$ in (4.6). Hence (4.16) agrees with (4.8).

Although we cannot write down the tensor structure of spacetime indices i, j for the general term $A^{(n)}$, we can determine the form of product appearing in $A^{(n)}$. $A^{(n)}$ is associated with the integral with $n - 1$ δ -functions. This set of δ -functions determines the tree graph with n vertices and $n - 1$ links. All the vertices should be connected by this graph in order for $A^{(n)}$ to be a single σ -integral term. The integral associated with the graph G is given by

$$\begin{aligned} I_n(G) &= \prod_{a=1}^n \int_0^1 d\sigma_a \left[\prod_{L:\text{link}} \delta(\sigma_{L_i L_f}) - \prod_{L:\text{link}} (1 - \delta(\sigma_{L_i L_f})) \right] \exp \left(\frac{i}{2} \sum_{a < b} k_a \theta k_b \tau(\sigma_{ab}) \right) \\ &= -J_n(k_1, \dots, k_n) + \sum_{L:\text{link}} J_2(k_{L_i}, k_{L_f}) J_{n-1}(k_1, \dots, k_{L_i} + k_{L_f}, \dots, k_n) + \dots, \end{aligned} \quad (4.20)$$

where $L_i, L_f \in \{1, \dots, n\}$ are the end points of L .

4.3. Area Derivative of Open Wilson Line

Using the boundary state picture of open Wilson lines, we can easily construct the field strength operator with open Wilson line attached. This operator was used in [19] to construct the Seiberg-Witten map in the form $A = A(\hat{A})$. The open Wilson line in the commutative and noncommutative picture is written as

$$\begin{aligned} \langle P(\sigma) | B \rangle &= \int [dx] \exp \left(i \int_0^1 d\sigma \frac{1}{2} x^i B_{ij} \partial_\sigma x^j - P_i(\sigma) \phi^i(x(\sigma)) \right) \\ &= \int [dx] \exp \left(i \int_0^1 d\sigma \frac{1}{2} x^i B_{ij} \partial_\sigma x^j + A_i(x) \partial_\sigma x^i - P_i(\sigma) x^i(\sigma) \right), \end{aligned} \quad (4.21)$$

where

$$\phi^i = x^i + \theta^{ij} \hat{A}_j. \quad (4.22)$$

By performing the “area derivative” of open Wilson line defined by

$$\frac{\delta}{\delta \Sigma^{ij}} = \lim_{\epsilon \rightarrow +0} \frac{\delta}{\delta P^i(\epsilon)} \frac{\delta}{\delta P^j(0)} - \frac{\delta}{\delta P^i(0)} \frac{\delta}{\delta P^j(\epsilon)}, \quad (4.23)$$

we find that

$$\text{Tr}_B \left([\phi^i, \phi^i] e^{-ik\phi} \right)_{\star_B} = \text{Tr}_{\mathcal{F}} \left([x^i, x^i] e^{-ikx} \right)_{\star_{\mathcal{F}}}. \quad (4.24)$$

Here \star_B and Tr_B are the star product and the trace defined by the symplectic form B . In the commutative side, the symplectic form becomes

$$\mathcal{F} = B + F = \theta^{-1} + F. \quad (4.25)$$

As discussed in [13], this relation (4.24) can be understood as the equivalence of star products:

$$Tf \star_B Tg = T(f \star_{\mathcal{F}} g). \quad (4.26)$$

With this operation T , ϕ and x are related by $\phi^i = T(x^i)$. See [23] for the derivation of the Seiberg-Witten map from the equivalence of star products.

The left-hand-side of (4.24) can be evaluated as

$$\text{Tr}_B \left([\phi^i, \phi^j] e^{-ik\phi} \right)_{\star_B} = i \text{Tr}_B \left[(\theta - \theta \widehat{F} \theta)^{ij} \mathbf{P} \exp \left(i \int_0^1 d\sigma \widehat{A}_i(x + \theta k \sigma) \theta^{ij} k_j \right) e^{-ikx} \right]_{\star_B}. \quad (4.27)$$

The right-hand-side of (4.24) is a complicated function of F , which can be calculated in principle by the formula in [24,25]. If we neglect the derivative of F , the commutator of x can be written as

$$[x^i, x^j]_{\star_{\mathcal{F}}} = i(\mathcal{F}^{-1})^{ij} + \mathcal{O}(\partial F) = i \left(\frac{1}{\theta^{-1} + F} \right)^{ij} + \mathcal{O}(\partial F). \quad (4.28)$$

5. Conclusion and Discussion

In this paper, we considered the open Wilson line as a momentum representation of the boundary state in a constant B -field background. The straight open Wilson line corresponds to the tachyonic particle state of closed string, and the curved open Wilson line appears as the coupling between a D -brane and an extended closed string. The generalized star products can be understood as the correlation functions of exponential operators on the boundary of worldsheet. The Seiberg-Witten map can be obtained by comparing the open Wilson line written in commutative and noncommutative pictures. The products of fields appearing in the Seiberg-Witten map are the combination of the generalized star products, and in principle the general term of the map can be determined by the rule of Wick contraction.

In [13], a closed form for the expression of the ordinary field strength in terms of the noncommutative gauge field was proposed. Although (4.24) gives the exact relation between F and \widehat{A} , it seems difficult to prove eq.(1.7) in [13].

In [9], it was shown that the interactions between the noncommutative gauge field and closed string modes in the superstring theory are different from those in the bosonic string theory. It is interesting to extend our result to the case of superstring.

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